International Journal of Mathematics and Statistics Invention (IJMSI)
e-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 13 Issue 5 || Sep. – Oct. 2025 || PP: 37-49

D – Valent Class of Functions with Ruscheweyh Derivatives –II

D. WILLIAM JOHN VICTOR
Associate Professor
Department of Mathematics
A. M. A. L CCOLLEGE , Anakapalli

Abstract

Some proeperties of the subclasses of S_n^p *(A, B)and* k_n^p *(A, B) of p-valent analytic functionswith negative coefficients by using the nth order Rushceweyh derivatives have been studied.*

I. INTRODUCTION

Let B(p) denote the class of functions.

$$f(z) = a_p z^p + \sum_{m=p+1}^{\infty} a_m z^m \text{ , } a_p > 0 \ a_m = 0,\ p = 1,$$

which are regular in the open unit disc E= {z :|z|<1}.

Let

$$\text{get } (z) = b_p z^p + \sum_{m=p+1}^{\infty} b_m z^m \quad b_p > 0 \text{ , } b_m \geq 0,\ p \geq 1,$$

belong to the class B(p). The Hadamard Product or convolution product. Which is denoted by *, of two functions f(z) and g(z) is defined by

$$(f * g) (z) = a_p b_p z^p + \sum_{m=p+1}^{\infty} a_m b_m z^m, \ z \in E$$

For $f \in B$ (p), define

$$D^{\alpha+p-1} f(z) = \frac{z^p}{(1-z)^{\alpha+9}} * f(z)$$

Where α is an integer greater than or equal to – p

Then

$$D^{n+p-1} f(z) = \frac{z(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!}$$

This definition is due to Ming-Po Chen and Ih-Ren-Lan [4].

When p = 1, Ruscheweyh [8] observed that.

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}$$

Where $n \in N]$ U{o}, N] set of natural numbers.

Let T be the subclass of B(p) consisting functions of the form

$$f(z) = a_p z^p - \sum_{m=p+1}^{\infty} a_m z_p^m, \quad a_m \geq 0,\ a_p > 0,\ p \geq 1$$

regular in the open unit disc E = {z : |z|<1}.

Let us define the quasi – Hadamard product of two functions f(z) and g(z) in the class T by

$$(f_* g)(z) = a_p b_p z^p = \sum_{m=p+1}^{\infty} a_m b_m z^m$$

$$\textit{Where } g(z) = b_p z^p - \sum_{m=p+1}^{\infty} b_m z^m,\ b_m \geq 0,\ p \geq 1.$$

In similar way K.Vinod Kumar [11] defined the quasi – Hadamard product of more than two functions.

Let $S_n^p(A,B)$ *denote the class of functions* $f \in T$ *such that*

$$\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - p\frac{1+Aw(z)}{1+Bw(z)}, -1 < A < B < 1, \ - \ \mathrm{p}\frac{1+Aw(z)}{1+Bw(z)},$$

Where $w \in H$ = {w(z) analytic, with w(o) = o and |w(z)|<1, z ϵ E}.

Let $K_n^p(\mathrm{A,B})$ denote the class of functions $f \in T$ *such that*

$$\frac{zf'(z)}{p} \in K_n^p(\mathrm{A,B}).$$

For a given real number z_0 ($z_0 \neq 0$, $-1 < z_0 < 1$), let T_1 and T_2 be the

subclasses of T satisfying $f(z_0) = z_o^p$ *and* $f'(z_o) = \mathrm{p}z_o^{p-1}$ *respectively. Let* $S_1^p(z_0)$, $K_1^p(z_o)$

and $K_2^p(z_o)$ *be the subclasses of T defined as* $S_1^p(z_o) = S_n^p(\mathrm{A,B}) \bigcap \mathrm{T}_1$,

$K_1^p(z_o) = K_n^p(\mathrm{A,B}) \bigcap \mathrm{T}_1, 1 = 1, 2.$

Now we study the necessary and sufficient condition for functions

to be in $S_1^p(\mathrm{A,B}), \mathrm{K}_n^p(\mathrm{A,B})$ *and* $S_1^p(z_0)$ *and* $K_1^p(z_o)$,

1= 1, 2. We determine radius of convexity for the classes $\mathrm{S}_1^p(z_o)$, 1 = 1,2. Also closure theorems are proved for these subclasses.

If we give particular values to p, A, B, and Z_o we get the results obtained by silverman [9, 10], Gupta and Jain [2, 3], Owa [5], K.S.Padmanabhan and R.Manjini [6, 7] and G.Ashok Kumar and G.L.Reddy [1].

II. COEFFICIENT INEQUALITIES

We now introduce the following notation for brevity.

$$A_{m,n} = \frac{(m+n+p-2)!}{(n+p)!(m-1)!} \text{ so that } A_{p,n} = \frac{(n+2p-2)!}{(n+p)!(p-1)!}$$

$B_{m,n} = p^2 + (n-1)p - m - n + 1$ so that $B_{p,n} = P^2 + (n-2)p - n + 1)$

$D_{m,n} = B(m+n+p-1) - Ap(n+p)$ so that $D_{p,n} = B(n+2p-1) - Ap(n+p)$

$C_{m,n} = D_{m,n} - B_{m,n}$ so that $C_{p,n} = D_{p,n} - B_{p,n}$

$E_m = A_{m,n} C_{m,n} - A_{p,n} C_{p,n} z_0^{m-p}$.

Theorem 2.1 – A function f(z) ∈ T is in $S_n^p(A,B)$ if and only if

$$\sum_{m=p+1}^{\infty} a_m A_{m,n} C_{m,n} \leq A_{p,n} C_{p,n} a_p \text{ where } n \in N]\, U\{0\} \text{ and} \quad \text{.... (2.1)}$$

$$-1 \leq A < \frac{B(n+2p-1)}{(n+p)p} \leq 1.$$

Proof : Suppose $f \in S_n^p(A,B)$. *Then*

$$\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} = P\frac{1+Aw(z)}{1+Bw(z)},\ -1 \leq A ,< B \leq 1, w(z) \in H,\ z \in E.$$

From this we get

$$w(z) = \frac{pD^{n+p-1} f(z) - D^{n+p} f(z)}{BD^{n+p} f(z) - ApD^{n+p-1} f(z)}$$

and | w(z)| < 1 implies that

$$|w(z)| = \left| \frac{pD^{n+p--1} f(z) - D^{n+p} f(z)}{BD^{n+p} f(z) - ApD^{n+p-1} f(z)} \right| < 1$$

Therefore

$$|w(z)| = \left| \frac{A_{p,n}B_{p,n}a_p z^P - \sum_{m=p+1}^{\infty} A_{m,n}B_{m,n}a_m z^m}{A_{p,n}D_{p,n}a_p z^P - \sum_{m=p+1}^{\infty} A_{m,n}D_{m,n}a_m z^m} \right| < 1$$

Or

$$\text{Re}\left\{ \frac{A_{p,n}B_{p,n}a_p - \sum_{m=p+1}^{\infty} A_{m,n}B_{m,n}a_m z^{m-p}}{A_{p,n}D_{p,n}a_p - \sum_{m=p+1}^{\infty} A_{m,n}D_{m,n}a_m z^{m-p}} \right\} < 1 \quad \ldots(2.3)$$

Consider the real values of z and z = r for O ≤ r < 1. Then for r = 0, the denominator of (2.3) is positive provided that.

$-1 \le A < \frac{B(n+2p-1)}{(n+p)p} \le 1$ for all r with 0 ≤ – <, since *(z) analytic in |z| <1. Then inequality

(2.3)

$$\sum_{m=p+1}^{\infty} A_{m,n}(D_{m,n} - B_{m,n})a_m r^{m-1} < A_{p,n}(D_{p,n} - B_{p,n})$$

Letting $r \rightarrow 1$, we get (2.1).

Conversely, suppose $f \in T$ and satisfies (2.1). for |z| = r, 0 ≤ r <1 and $-1 \le A < \frac{B(n+2p-1)}{(n+p)p} \le 1$. We have (2.4) by (2.1) Since $r^{m-1} < 1$. So we have.

$$\left| A_{p,n}B_{p,n}a_p - \sum_{m=p+1}^{\infty} A_{m,n}B_{m,n}a_m z^{m-1} \right| \le A_{p,n}, B_{p,n}a_p - \sum_{m=p+1}^{\infty} A_{n,n}B_{n,n} \in_m z^{m-1} \Big|$$

$$< Ap,n\ Dp,n\ Ap - \sum_{m=p+1}^{\infty} A_{m,n}D_{m,n}a_m r^{m-1} \le \Big| A_{p,n}D_{p,n}a_p, - \sum_{m=p+1}^{\infty} A_{m,n}D_{m,n}a_m z^{m-1}$$

Which gives (2.2) and hence follows that.

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{1+Aw(z)}{1+Bw(z)},\ w \in H,\ z \in E, -1 \le A < \frac{B(n+2p-1)}{(n+p)p} \le 1$$

Which implies that f ∈ $S_n^p(A,B)$.

Remark The result is not true for the general class $S_n^p(A,B)$.

Where $-1 \le A < B \le 1$. However we have proved it only for a restricted class of it.

Corollary 2.1 –A function f(z) ∈ T is in $S_n^p(A,B)$.

$$-1 \le A < \frac{B(n+2p-1)}{(n+p)p} \le 1\text{, then}$$

$$a_m \le \frac{A_{p,n}C_{p,n}a_p}{A_{m,n}C_{m,n}},\ m \ge p+1.$$

Proof is direct consequence of theorem 2.1.

<u>**Theorem 2.2**</u> – A function f(z) ∈ T is in $K_n^p(A,B)$. if and only if

$$\sum_{m=p+1}^{\infty} \frac{m A_{m,n}C_{m,n}a_m}{p A_{p,n}C_{p,n}} \le a_p \text{and} -1 \le A < \frac{B(n+2p-1)}{(n+p)p} \le 1 \quad \text{.... (2.5)}$$

Proof follows by the definition of $K_n^p(A,B)$. and theorem 2.1.

Corollary 2.2 – A function f(z) ∈ T is in $K_n^p(A,B)$,

$$-1 \le A < \frac{B(n+2p-1)}{(n+p)p} \le \frac{pA_{p,n}C_{p,n}a_p}{mA_{m,n}C_{m,n}},m \ge p+1.$$

Proof is obvious by theorem 2.2.

Theorem 2.3 - A function f(z) ∈ T_1 is in $S_1^p(z_o)$ if and only if

$$\sum_{m=p+1}^{\infty} \frac{E_m a_m}{A_{p,n}C_{p,n}} \le 1. \quad 2.6$$

Proof. Let f(z) ∈ T_1 is in $S_1^p(z_o)$. Then z_o be a fixed real number such that $-1 < z_o < 1 \ and \ z_o \ne 0,\ f(z_o) - a_p z_o^p - \sum_{m=p+1}^{\infty} a_m z_o^m$

We have $a_p = 1 + \sum_{m=p+1}^{\infty} a_m z_o^{m-p}$ since we have taken f(z_o) = z_o^p

By definition of $S_1^p(z_o)$ we have $f \in S_n^p(A,B),\ therefore\, by$

Theorem 2.1 and using the fact that $a_p = 1 + \sum_{m=p+1}^{\infty} a_m z_o^{m-p}$, we get

$$\sum_{m=p+1}^{\infty} a_m A_{m,n} C_{m,n} \le A_{p,n} C_{p,n} \ (1+ \sum_{m=p+1}^{\infty} a_m z_o^{m-p})$$

1.8. $$\sum_{m=p+1}^{\infty} (A_{m,n} C_{m,n,} - A_{p,n} C_{p,n}) \, a_m \le A_{p,n} C_{p,n}$$

1.9. $$\sum_{m=p+1}^{\infty} \frac{E_m a_m}{A_{p,n} C_{p,n}} \le 1.$$

Conversely, Suppose that f(z) ∈ T₁ And function satisfies the eqn. (2.6.)

By the relation f(z$_o$) = Z_o^p , we get $\sum_{m=p+1}^{\infty} a_m z_o^{m-1} = a_p - 1$.

Substituting for (a$_p$-1) in (2.6) we get (2.1) therefore it

Belongs to the classes $S_n^p(A,B)$ and by hypothesis f ∈ T₁ . Hence

$$f \in S_n^p(A,B) \cap T_1 = S_1^p(z_o)$$

Corollary 2.3 - if $f \in S_1^p(z_o)$ then $a_m \le \frac{A_{p,n} C_{p,n}}{E_m}, \ for \ m \ge p+1$

With extremal function f(z) $= \frac{A_{m,n} C_{m,n} z^p - A_{p,n} C_{p,n} z^m}{E_m}, \ m \ge p+1$.

Proof follows from theorem 2.3.

Theorem 2.4- let $f \in T_1$. Then f ∈ $K_1^p \ (z_o)$ fin and only if

$$\sum_{m=p+1}^{\infty} \{\frac{\frac{m}{p} A_{m,n} C_{m,n} - A_{p,n} C_{p,n} z^{m-p}{}_o}{A_{p,n} C_{p,n}}\} a_m \le 1.$$

Proof follows as in theorem 2.2.

Theorem 2.5 – let $f \in T_2$. Then $f \in S_2^p \ (z_o)$ if and only if

$$\sum_{m=p+1}^{\infty} \{\frac{A_{m,n} C_{m,n} - \frac{m}{p} E m z^{m-p}{}_o}{r!(n-r)!}\} a_m \le 1. \qquad \text{…(2.7)}$$

Proof : Let $f \in S_2^p \ (z_o)$. Tehn for fixed real number Z$_o$ such that

$Z_o \neq 0$, $-1 < z_o < 1$, $f'(z_o) = pa_pz_o^{p-1} - \sum_{m=p+1}^{\infty} ma_mz_o^{m-1}$. since

$\frac{f'(z_o)}{pz_o^{p-1}} = 1,$ we have $a_p = 1 + \sum_{m=p+1}^{\infty} \frac{m}{p} a_m z_o^{m-p}$. $f \in S_2^p(z_o)$ *implies*

That $f \in S_n^p(A,B)$ and so theorem 2.1 holds for f and hence by Substituting the value of a_p in (2.1) we get (2.7).

Conversely, let $f \in T_2$ and (2.7) hold. Since.

$f'(Z_o) = pz_o^{p-1}$ we have $\sum_{m=p+1}^{\infty} \frac{m}{p} a_m z_o^{m-p} = a_p - 1.$ substituting

the value of $\sum_{m=p+1}^{\infty} \frac{m}{p} a_m z_o^{m-p}$ in (2.7), we get (2.1) from which

we conclude that $f \in S_n^p(A,B)$ by theorem 2.1. hence $f \in S_2^p(z_o)$.

Theorem 2.6 Let $f \in T_2$. Then $f \in K_2^p(z_o)$ if and only if

$$\sum_{m=p+1}^{\infty} \frac{mE_m}{pA_{p,n}C_{p,n}} \leq 1.$$

Proof follows as in theorem 2.5.

3. THE RADIUS OF p-VALENT CONVEXITY
OF THE CLASSES $S_1^P(z_o),\ 1=1,2.$

Theorem 3.1 - Let $f \in T.$ *if* $f \in S_1^p(z_o)$ *or* $S_2^p(z_o)$, *then* f *is* p-valently convex in the disc $|z|<r$ where

$$r = \inf_m \left[\frac{p2A_{m,n}C_{m,n}}{m2A_{p,n}C_{p,n}}\right]^{1/(m-p)}$$

The bound is sharp.

Proof : To prove the theorem, it is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)} - p\right| \leq p \ \ for \ \ |z| < r,\ z \in E.$$

Now we have

$$\left|\frac{(1=p)f'(z)=zf''(z)}{f'(z)}\right|=\left|\frac{=\sum\limits_{m=p+1}^{\infty} m(m-p)\,a_m z^{m-1}}{pa_p z^{p-1} = \sum\limits_{m=p+1}^{\infty} m a_m z^{m-1}}\right|.$$

$$\le \frac{\sum\limits_{m=p+1}^{\infty}\frac{m}{p}(m-p)a_m|z|^{m-p}}{a_p = \sum\limits_{m=p+1}^{\infty}\frac{m}{p}a_m|z|^{m-p}}$$

Consider the values of z for which

$$|z| \le \inf_m \left\{(p/m)2\,\frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}}\right\}^{1/(m-p)}$$

So that $|z|m-p\le \frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}}(p/m)^2\ holds.$

Then $\sum\limits_{m=p+1}^{\infty}(m/p)a_m|z|^{m-p} \le \sum\limits_{m=p+1}^{\infty}(m/p)(p/m)\,2\,\frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}}a_m.$

Now $\sum\limits_{m=p+1}^{\infty}(m/p)a_m|z|^{m-p} \le a_p\ provided\ \sum\limits_{m=p+1}^{\infty}\frac{PA_{m,n}C_{m,n}}{mA_{p,n}C_{p,n}}a_m < a_p.$

Which holds if $\sum\limits_{m=p+1}^{\infty}\frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}}a_m < a_p = 1 + \sum\limits_{m=p+1}^{\infty}a_m\,z_o^{m-p}$

This is equivalent to $\sum\limits_{m=p+1}^{\infty}\frac{E_m a_m}{A_{p,n}C_{p,n}}$, which is true by

Theorem 2.2 or theorem 2.4. Hence we can rewrite the denominator of the right hand side of inequality (3.1) for considered values of z, using the fact that

$$a_p > \sum_{m=p+1}^{\infty}(m/p)a_m\ z^{m-p}$$

Thus

$$\left|\frac{(1-p)zf'(z)+zf''(z)}{f'(z)}\right| \le \sum_{m=p+1}^{\infty} \frac{(m/p)(m-p)\,a_m|z|^{m-p}}{a_p = \sum\limits_{m=p+1}^{\infty} a_m|z|^{m-p}} \le p \;\; if$$

$$\sum_{m=p+1}^{\infty} m(m-p)\,a_m|z|^{m-p} \le p^2 a_p = \sum_{m=p+1}^{\infty} pma_m|z|^{m-p} \quad ...(3.2)$$

$if \;\; f \in S_1^p(zo), (3.2)\, is\, equivalent\, to$

$$\sum_{m=p+1}^{\infty} a_m\,|z|^{m-p} \le a_p = 1 + \sum_{m=p+1}^{\infty} a_m z_o^{m-p}$$

That is

$$\sum_{m=p+1}^{\infty} ((m/p)^2|z|^{m-p} - z_o^{m-p})a_m \le 1 \qquad \ldots (3.3)$$

Again if $f \in S_2^p(zo),\ (3.2)\, is\, equivalent\, to$

$$\sum_{m=p+1}^{\infty} ((m/p)^2|z|^{m-p} - (m/p)\,z_o^{m-p})a_m \le 1 \qquad \ldots (3.4)$$

In view of theorem 2.2 $f \in S_1^p(z_o)$ if and only if

$$\sum_{m=p+1}^{\infty} \left(\frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}} - z_o^{m-p}\right) a_m \le 1$$

Hence inequality (3.3) is true if

$(m/p)^2\,|z|^{m-p} - Z_o^{m-p} = \dfrac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}} - z_o^{m-p} \;, for\ all\ m \ge p+1$

That is, if

$$|z| \le \left[((p/m)^2 \frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}}) \right]^{1/(m-p)} \qquad for\ all\, m \ge$$

Again in view of theorem 2.4, $f \in S_2^p(z_o)$ *if and only if*

$$\sum_{m=p+1}^{\infty}\left(\frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}} - \frac{m}{p} z_o^{m-p}\right) a_m \le 1$$

Inequality (3.4) is true if

$$(m/p)^2 |z|^{m-p} - (m/p) z_o^{m-p} \le \frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}} - (m/p) z_o^{m-p}, \text{ for all } m \ge p+1$$

Then result is sharp with the extremal function

$$f_m(z) = \frac{A_{m,n}C_{m,n}z^p - A_{p,n}C_{p,n}z^m}{A_{p,n}C_{p,n}},\ m \ge p+1.$$

Remark : The conclusion of theorem 3.1 is independent of the point Z_o.

IV. CLOSURE THEOREMS

In this section we prove that the classes $S_1^p\left(z_o\right)$ and $K_1^p\left(z_o\right)$, i = 11,2 are closed under convex linear combination and also show that the functions of these classes can be expressed in a particular form.

Theorem 4.1 – The class $S_1^p\left(z_o\right)$ is closed under convex linear combination.

Proof : Let the functions f(z) = $a_p z^p$ - $\sum_{m=p+1}^{\infty} a_m z^m,\ \left(a_m \ge o, a_p > o\right)$ and g(z) = $b_p z^p$ - $\sum_{m=p+1}^{\infty} b_m z^m,\ \left(b_m \ge o, b_p > o\right)$ be any two functions of the class $S_1^p\left(z_o\right)$. For λ Such that $0 \le \lambda \le 1$, it suffices to show that h(z) = $(1-\lambda)f(z) + \lambda g(z),\ z \in E,$ is also a function of $S_1^p\left(z_o\right)$.

Now

$$h(z) = ((1-\lambda)a_p = \lambda b_p) z^p - \sum_{m=p+1}^{\infty} ((1-\lambda)a_m + \lambda b_m) z^m.$$

Applying theorem 2.3 to f,g $\in S_1^p(z_o)$, we have

$$\sum_{m=p+1}^{\infty} E_m((1-\lambda)\,a_m+\lambda b_m = (1-\lambda)\sum_{m=p+1}^{\infty} E_m a_m + \lambda \sum_{m=p+1}^{\infty} E_m b_m$$

$$\le (1-\lambda)\,A_{p,n}C_{p,n} + \lambda\, A_{p,n}C_{p,n} = A_{p,n}C_{p,n}.$$

Also h(z$_o$) = $(1-\lambda)\, f(z_o)+\lambda g(z_o) = (1-\lambda)z_o^p + \lambda\, z_o^p = z_o^p$

Which implies that h is in T$_1$. Hence by theorem 2.3 h belongs to $S_1^p\left(z_o\right)$.

Theorem 4.2 - $S_2^p\left(z_o\right)$ is closed under convex linear combination.

Proof follows as in theorem 4.1.

Theorem 4.3 - $K_1^p\left(z_o\right)$, i = 1,2, are closed under convex linear combination.

Theorem 4.4 –Define $f_p(z) = Z^p$ and $f_m(z) = \dfrac{A_{m,n}C_{m,n}z^p - A_{p,n}C_{p,n}z^m}{E_m}$,

$$m \ge p+1.$$

Then $f \in S_1^p(z_o)$ if and only iff is of the form f(z) = $\sum_{m=p+1}^{\infty} \lambda_m f_m(z)$.

$$z \in E \; where\, \lambda_m \ge 0 \; for\; m \ge p \; and \sum_{m=p}^{\infty} \lambda_m = 1.$$

Proof : Suppose f(z) = $\sum_{m=p}^{\infty} \lambda_m f_m(z) = \lambda_p f_p(z) + \sum_{m=p+1}^{\infty} \lambda_m f_m(z)$

$$= \lambda_p z_p + \sum_{m=p+1}^{\infty} \frac{A_{m,n}C_{m,n}z^p - A_{p,n}C_{p,n}z^m}{E_m}\lambda_m$$

$$= (\lambda_p + \sum_{m=p+1}^{\infty} \frac{A_{m,n}C_{m,n}}{E_m})z^p - \sum_{m=p+1}^{\infty} \frac{A_{p,n}C_{p,n}}{E_m}\lambda_m z^{m.}$$

Since $\lambda_p \ge 0.$

Further $f_m(z_o) = Z_o^p$. Therefore

$$f(z_o) = \sum_{m=p}^{\infty} \lambda_m f_m(zo) = \sum_{m=p}^{\infty} \lambda_m z_o^p = z_o^p \sum_{m=p}^{\infty} \lambda_m = z_o^p.$$

This proves that $f \in T_1$. Hence by Theorem 2.3, $f \in S_1^p(z_o)$.

Conversely, Let $f \in S_1^p(z_o)$ then $a_p = 1 + \sum_{m=p+1}^{\infty} a_m z_o^{m-p}$

Setting $\lambda_m = \frac{E_m}{A_{p,n}C_{p,n}} a_m,\ m \geq p+1 \, and \, \lambda_p = \sum_{m=p+1}^{\infty} \frac{A_{m,n}C_{m,n}}{A_{p,n}C_{p,n}} a_m.$

We note that $\lambda p \geq 0$ by theorem 2.1. Now we have.

$$f(z) = a_p z^p - \sum_{m=p+1}^{\infty} a_m z^m = \lambda_p z^p + \sum_{m=p+1}^{\infty} \lambda_m (1 + (z_o^{m-p} - z^{m-p})(a_m / \lambda_m)) z^p$$

$$= \lambda_p z^p + \sum_{m=p+1}^{\infty} \frac{A_{m,n}C_{m,n}z^p - A_{p,n}C_{p,n}}{E_m} \lambda_m$$

$$= \lambda_p z^p + \sum_{m=p+1}^{\infty} \lambda_m f_m(z) = \sum_{m=p}^{\infty} \lambda_m f_m(z).$$

Hence the theorem.

We can prove the following results in a similar manner.

Theorem 4.5 – Define $f_p(z) = z^p$ and

$$f_m(z) = \frac{A_{m,n}C_{m,n}z^p - A_{p,n}C_{P,n}z^m}{A_{m,n}C_{m,n} - (m/p)\, A_{p,n}C_{p,n}z_o^{m-p}},\ m \geq p+1.$$

Then $f \in S_2^p(z_o)$ if and only if f is of the form $f(z) = \sum_{m=p}^{\infty} \lambda_m f_m(z)$,

$Z \in E$ where $\lambda_m \geq 0$ for $m \geq p$ and $\sum\limits_{m=p}^{\infty} \lambda_m = 1$.

Theorem 4.6 - Define $f_p(z) = z^p$ and

$$f_m(z) = \frac{mA_{m,n}C_{m,n}z^p - pA_{p,n}C_{p,n}z^m}{mA_{m,n}C_{m,n} - pA_{p,n}C_{p,n}z_o^{m-p}}, \; m \geq p+1, p \geq 1.$$

Then $f \in K_1^p(z_o)$ if and only if f is of the form

$$f(z) = \sum_{m=p}^{\infty} \lambda_m f_m(z), \; Z \in E \text{ where } \lambda_m \geq 0 \text{ for } m \geq p \text{ and } \sum_{m=p}^{\infty} \lambda_m = 1.$$

Theorem 4.7 – Define $f_p(z) = z^p$ and

$$f_m(z) = \frac{A_{m,n}C_{m,n}z^p - (p/m)A_{p,n}C_{p,n}z^m}{E_m}, \; m \geq p+1, \; p \geq 1.$$

Then $f \in K_2^p(z_o)$ if and only if f is of the form

$$f(z) = \sum_{m=p}^{\infty} \lambda_m f_m(z), \; z \in E \text{ where } \lambda_m \geq 0 \text{ for } m \geq p \text{ and } \sum_{m=p}^{\infty} \lambda_m = 1.$$

REFERENCES

[1]. V.P.Gupta and P.K.Jain, "Certain Classes of univalent functions with negative coefficients", Bull. Austral. Mat. Soc. 14(1976). 409-16.

[2]. V.P.Gupta and P.K.Jain, "Certain Classes of univalent functions with negative coefficients'. Bull. Asutral. Math. Soc. 15(1976). 467-73.

[3]. Ming- Po-Chen and In-Ren Lan, "on certain inequalities for some regular functions defined on the unit disk", Bull. Austral. Math. Soo. 35(1987).

[4]. S.Owa, " on new classes of analytic functions with negative coefficients", Internet .J.Math. Sci. and Math. Sci.7(1984). 719-30.

[5]. K.S.Padmanabhan, and R.Manjini, " Certain Classes of Analytic functions with negative coefficients", Indian J.Pure and Appl.Maths. 17(10) (1986), 1210-1223.

[6]. K.S.Padmanabhan and R.Mnjini, "Certain Classes of Analytic functions with negative coefficients – II" Indian J.Pure and Appl.Math. 18(2) (1987), 159-172.

[7]. S.Ruscheweyh, "New Criteria for univalent function", Proc. Amer.Math. Soc. 49(1975), 109-15.

[8]. H.Silverman, "Univalent functions with negative Coefficients", Proc. Amer.Math.Soc.51(1975), 109-16.

[9]. H.Silverman, "Extreme points of univalent functions with two fixed points", Trans.Amer.Math. Soc.219 (1976), 387-395.

[10]. Vinod Kumar, "Quasi – Hadamard Product of certain Univalent functions", Journal of Mathematical Analysis and Applications, 126(1987), 70-77.