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Three Dimensional Generalized Arithmetic Progressions of Multiplicity Two

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Abstract
The prime aim of presenting this article is to review, expose and introduce the extended concept of arithmetic progressions combined with the concept of dimension (three) and multiplicity (two) applied on common differences, which was previously published and discussed in chapter five in two books on multidimensional arithmetic progressions properly cited in the reference and article. It presents a discussion about different basic properties of three dimensional generalized arithmetic progressions with multiplicity two. It discusses the way to find the general term, sum of first finite number of terms and the working rules to find the arithmetic means between any two arbitrary terms of such generalized progressions. The mathematics teachers will find it useful to introduce the topic with different look and a research oriented approach to teaching. The article also opens a new scope of research and its applications.

Keywords: *arithmetic progression, generalized arithmetic progression, common difference, arithmetic mean, dimension, multiplicity, sum of terms, etc.*
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I. Introduction:

We know that the study of patterns leads to many significant generalizations. The succession of numbers (real or complex) of which one number is designated as the first, another as the second, another next as the third and so on gives rise to what we call a sequence. It has been found that Babylonians, 4000 years ago, knew about arithmetic and geometric sequences. According to Boethius (510 AD), arithmetic, geometric and harmonic sequences were known to early Greek writers (James & James, 2001; Katz, 2019; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026).

A sequence states a number of things that occur one after another. It is the order in which a number of things are written. It is a collection of numbers (may be repeated or different) in a specific order followed by a definite rule. The members of the sequence are called elements or terms or members itself. We know that in a set we don't repeat elements but in a sequence it may be repeated multiple times and at different positions. In it the order of the terms matter but in a set order doesn't matter. It is generally defined by a function from an arbitrary index set. It can be finite or infinite depending on the number of terms. A sequence is generally denoted by its nth term, where n denotes the nth element of the sequence from beginning (James & James, 2001; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026).

There are many ways to denote a sequence, one of which is to list all its members like the first five even numbers form the sequence {2, 4, 6, 8, 10}. This notation is also used to denote infinite sequences like the infinite sequence of natural numbers $\{1, 2, 3, \dots\}$. Here three dots "…" denote that the terms are continued till infinite terms. As discussed earlier we can represent the finite sequence having its nth term as t_n by $(t_n)_{n=1}^{r}$, where r is a finite natural number like 10, 15, 50, etc. The infinite sequence for the same nth term of the sequence is denoted by $(t_n)_{n=1}^{\infty}$, where n is a natural number (James & James, 2001; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026).

Thus formally a sequence is defined as a function whose domain is an interval of integers and the values taken by the function are the elements of the sequence. The possible values of the function i.e. co-domain of the function are the terms of the sequence and are generally real numbers and our study is also limited to the real numbers (James & James, 2001; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026).

The sum of the terms of a sequence is called a series and is written as $\sum_{n=1}^{\infty} t_n = t_1 + t_2 + t_3 + \cdots$, where t_n is the nth term of the sequence of real or complex numbers. The partial sum of a series is the expression obtained by replacing the infinity symbol with a finite number (say N) as $\sum_{n=1}^{N} t_n = t_1 + t_2 + t_3 + \ldots + tN$. Generally it is denoted by SN. If the sequence of the partial sum converges, we say that the infinite series is convergent (Convergent series - Wikipedia contributors, 2025; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Series (mathematics) - Wikipedia contributors, 2026; Ulas, 2005; Yadav, 2008a, 2008b, 2010, 2020, 2026). The general expression for the sum of infinite geometric series was given by Frenchman Francois-Vieta (1540 - 1603 AD). In 1671 AD James Gregory used the term infinite series in connection with infinite sequence (James & James, 2001; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026).

A sequence is also known as progression (Progression - Wikipedia contributors, 2025). It may be divided into many parts out of which the one is the arithmetic progression (Arithmetic progression - Wikipedia contributors, 2026; Dawson, 2012; Progression - Wikipedia contributors, 2025). It is a sequence of numbers such that the difference from any succeeding term to its preceding term remains a fixed constant throughout the entire sequence. The fixed difference is known as common difference of the arithmetic progression. In it we study about finding the genera term, particular term, sum of the first n and infinite terms and some properties (Arithmetic progression - Wikipedia contributors, 2026; Convergent series - Wikipedia contributors, 2025; Progression - Wikipedia contributors, 2025; Series (mathematics) - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026). A finite part of an arithmetic progression is known as a finite arithmetic progression or generally arithmetic progression. The sum of a finite arithmetic progression is called an arithmetic series (Arithmetic progression - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026).

II. Preliminary Ideas:

In our discussion, we should know the basic concepts of the following mathematical terms:

2.1. Multiplicity: As have been discussed that in a sequence an element can appear multiple times i.e. multiplicity defines that how many times an element appears in any mathematical calculations or in a set. If an element occurs more than once in a set, it is referred as a multiset (Multiplicity (mathematics) - Wikipedia contributors, 2025; Sarkar, 2003; Yadav, 2008a, 2008b, 2010, 2020, 2026). This concept has been applied in the arithmetic progression with the common difference, as will be seen soon. In physics and mathematics, the dimension of a space or object is defined as the minimum number of coordinates needed to specify any point in it (Dimension - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026). In our work, we have taken dimension in a different meaning as will be discussed in the paper. Before going to discuss the proposed title and its related properties, let us have some brief ideas about the following terms:

2.2. Generalized Arithmetic Progression: A multiple arithmetic progression, generalized arithmetic progression, or k-dimensional arithmetic progression, is a set of integers constructed as an arithmetic progression, but allowing several k possible common differences (Generalized arithmetic progression - Wikipedia contributors, 2024; Yadav, 2008a, 2008b, 2010, 2020, 2026). The number *k* that is the number of permissible common differences is called the *dimension* of the generalized arithmetic progression.

2.3. K-Dimensional Generalized Arithmetic Progression: A multiple arithmetic progression / generalized arithmetic progression / k-dimensional arithmetic progression / k-dimensional generalized arithmetic progression is a set of numbers constructed as an arithmetic progression, but allowing k possible common differences one by one periodically. The number *k* the number of permissible common differences is called the *dimension* of the generalized arithmetic progression (Generalized arithmetic progression - Wikipedia contributors, 2024; Yadav, 2008a, 2008b, 2010, 2020, 2026).

2.4. K-Dimensional Generalized Arithmetic Progression with Multiplicity m: If in k- dimensional arithmetic progression, every common difference is applied m times successively at a time and periodically, then we get a k-D.A.P. with multiplicity m. Here the number of times (m) a common difference is applied has been named as *multiplicity* of the k-D.G.A.P. In the present article we have denoted it by k-DGAP(m) or by k-DAP(m) (Generalized arithmetic progression - Wikipedia contributors, 2024; Yadav, 2008a, 2008b, 2010, 2020, 2026).

III. Discussion:

Yadav (2008a, 2008b, 2010, 2020, 2026) has discussed two dimensional generalized arithmetic progression with multiplicity one, two dimensional generalized arithmetic progression with multiplicity two and three dimensional generalized arithmetic progression with multiplicity one with their general classical properties and examples. Yadav (2020, 2026) authored two books on it entitled Multidimensional Arithmetic Progressions, one with international edition and another with Indian edition and discussed the generalized arithmetic progressions with many important properties. In this article we are representing the concepts as follows:

3.1. Three Dimensional Arithmetic Progressions with Multiplicity Two: If a is the first term, d_1 is the first common difference, d_2 is the second common difference, and d_3 is the third common difference, then the general form of 3-D.A.P. with multiplicity two (may be denoted by 3-D.A.P.M.(2)) is given by

$$\begin{gathered} a \\ a + d_1 \\ a + 2d_1 \\ a + 2d_1 + d_2 \\ a + 2d_1 + 2d_2 \\ a + 2d_1 + 2d_2 + d_3 \\ a + 2d_1 + 2d_2 + 2d_3 \\ a + 3d_1 + 2d_2 + 2d_3 \\ a + 4d_1 + 2d_2 + 2d_3 \\ a + 4d_1 + 3d_2 + 2d_3 \\ a + 4d_1 + 4d_2 + 2d_3 \\ a + 4d_1 + 4d_2 + 3d_3 \\ a + 4d_1 + 4d_2 + 4d_3 \\ a + 5d_1 + 4d_2 + 4d_3 \\ a + 6d_1 + 4d_2 + 4d_3 \\ a + 6d_1 + 5d_2 + 4d_3 \\ a + 6d_1 + 6d_2 + 4d_3 \\ a + 6d_1 + 6d_2 + 5d_3 \\ a + 6d_1 + 6d_2 + 6d_3 \\ a + 7d_1 + 6d_2 + 6d_3 \end{gathered}$$

and so on.

Example 3.1: 2, 5, 8, 12, 16, 21, 26, 29, 32, 36, 40, 45, 50, 53, … is a 3-D.A.P.M.(2) with 2 as first term and 3, 4, 5 as successive common differences.

3.2. General Term of a 3-D.A.P.M.(2): The general term or nth term of 3-D.A.P.M.(2) cannot be written in a single mathematical expression. There arise six cases to find the nth term, as discussed below one by one: Let a be the first term, d_1, d_2, and d_3 be the three c.d.'s and multiplicity M = 2. Then

Case I: When n is of the form (6m–5), where m = 1, 2, 3, … i.e., n = 1, 7, 13, 19, 25, …

Then

$$t_n = t_{(6m-5)} = a + (m-1)\, 2\, (d_1 + d_2 + d_3)$$

$$\Rightarrow\ t_n = a + 2\left(\frac{n-1}{6}\right)(d_1 + d_2 + d_3)$$

Case II: When n is of the form (6m–4); where m = 1, 2, 3, … i.e., n = 2, 8, 14, 20, 26, … Then

$$t_n = t_{(6m-4)} = (a + d_1) + (m-1)\, 2\, (d_1 + d_2 + d_3)$$

$$\Rightarrow\ t_n = (a + d_1) + 2\left(\frac{n-2}{6}\right)(d_1 + d_2 + d_3)$$

Case III: When n is of the form (6m-3); where m = 1, 2, 3, … i.e., n = 3, 9, 15, 21, … Then

$$t_n = t_{(6m-3)} = (a + 2d_1) + (m-1)\, 2\, (d_1 + d_2 + d_3)$$

$$\Rightarrow\ t_n = (a + 2d_1) + 2\left(\frac{n-3}{6}\right)(d_1 + d_2 + d_3)$$

Case IV: When n is of the form (6m-2); where m = 1, 2, 3, … i.e., n = 4, 10, 16, 22, … Then

$$t_n = t_{(6m-2)} = (a + 2d_1 + d_2) + (m-1)\, 2\, (d_1 + d_2 + d_3)$$

$$\Rightarrow\ t_n = (a + 2d_1 + d_2) + 2\left(\frac{n-4}{6}\right)(d_1 + d_2 + d_3)$$

Case V: When n is of the form (6m-1); where m = 1, 2, 3, … i.e., n = 5, 11, 17, 23, … Then

$$t_n = t_{(6m-1)} = (a + 2d_1 + 2d_2) + (m - 1)\,2\,(d_1 + d_2 + d_3)$$

$$\Rightarrow t_n = (a + 2d_1 + 2d_2) + 2\left(\frac{n-5}{6}\right)(d_1 + d_2 + d_3)$$

Case VI: When n is of the form (6m); where m = 1, 2, 3, … i.e., n = 6, 12, 18, 24, … Then

$$t_n = t_{(6m)} = (a + 2d_1 + 2d_1 + d_3) + (m - 1)\,2\,(d_1 + d_2 + d_3)$$

$$\Rightarrow t_n = (a + 2d_1 + 2d_2 + d_3) + 2\left(\frac{n-6}{6}\right)(d_1 + d_2 + d_3)$$

Example 3.2: Find the 13th, 14th, 15th, 16th, 17th, and 18th terms of the sequence of 3-D.A.P.M.(2) with first term 2 and c.d.'s 3, 4, 5 respectively.

Here $a = 2$, $d_1 = 3$, $d_2 = 4$, $d_3 = 5$, and M = 2. Therefore

$$t_{13} = t_{18-5} = t_{6m-5}(m = 3) = a + 2\frac{n-1}{6}(d_1 + d_2 + d_3)$$
$$= 2 + 2\tfrac{13-1}{6}(3 + 4 + 5) = 50$$

$$t_{14} = t_{18-4} = t_{6m-4}(m = 3) = (a + d_1) + 2\frac{n-2}{6}(d_1 + d_2 + d_3)$$
$$= (2 + 3) + 2\tfrac{14-2}{6}(3 + 4 + 5) = 53$$

$$t_{15} = t_{18-3} = t_{6m-3}(m = 3) = (a + 2d_1) + 2\frac{n-3}{6}(d_1 + d_2 + d_3)$$
$$= (2 + 6) + 2\tfrac{15-3}{6}(3 + 4 + 5) = 56$$

$$t_{16} = t_{18-2} = t_{6m-2}(m = 3) = (a + 2d_1 + d_2) + 2\frac{n-4}{6}(d_1 + d_2 + d_3)$$
$$= (2 + 6 + 4) + 2\tfrac{16-4}{6}(3 + 4 + 5) = 60$$

$$t_{17} = t_{18-1} = t_{6m-1}(m = 3) = (a + 2d_1 + 2d_2) + 2\frac{n-5}{6}(d_1 + d_2 + d_3)$$
$$= (2 + 6 + 8) + 2\tfrac{17-5}{6}(3 + 4 + 5) = 64$$

$$t_{18} = t_{18} = t_{6m}(m = 3) = (a + 2d_1 + 2d_2 + d_3) + 2\frac{n-6}{6}(d_1 + d_2 + d_3)$$
$$= (2 + 6 + 8 + 5) + 2\tfrac{18-6}{6}(3 + 4 + 5) = 69$$

Hence the given 3-D.A.P.M.(2) is

2, 5, 8, 12, 16, 21, 26, 29, 32, 36, 40, 45, 50, 53, 56, 60, 64, 69, …

3.3. Sum of First n Terms of a 3–D.A.P.M.(2): The sum of all the terms of a 3–D.A.P. with multiplicity two is called a 3–D.A.P. Series with Multiplicity Two. Let a be the first term, d_1, d_2, and d_3 be three c.d.'s with multiplicity two and let S_n be the sum of first n terms, then

$$S_n = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + \dots\dots\dots + t_n$$

$= (t_1 + t_7 + t_{13} + t_{19} + \dots) + (t_2 + t_8 + t_{14} + t_{20} + \dots) + (t_3 + t_9 + t_{15} + t_{21} + \dots) +$
$\quad (t_4 + t_{10} + t_{16} + t_{22} + \dots) + (t_5 + t_{11} + t_{17} + t_{23} + \dots) + (t_6 + t_{12} + t_{18} + t_{24} + \dots)$
$= S_{6m-5} + S_{6m-4} + S_{6m-3} + S_{6m-2} + S_{6m-1} + S_{6m}$
$= S_A + S_B + S_C + S_D + S_E + S_F$

To find S_A, S_B, S_C, S_D, S_E, S_F, we consider six different cases as follows:

Case A: When only the terms n = 1, 7, 13, 19, 25, … of the form (6m-5) for m = 1, 2, 3, … are taken into consideration, then we get an arithmetic series with first term a and common difference $2(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_A = \frac{m}{2}[2a + (m-1)2(d_1 + d_2 + d_3)]$$
$$= \frac{n+5}{12}\left[2a + 2\left(\frac{n-1}{6}\right)(d_1 + d_2 + d_3)\right]$$

Case B: When only the terms n = 2, 8, 14, 20, 26, … of the form (6m-4) for m = 1, 2, 3, … are taken into consideration, then we get an arithmetic series with first term $(a+d_1)$ and common difference $2(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_B = \frac{m}{2}[2(a + d_1) + (m-1)2(d_1 + d_2 + d_3)]$$
$$= \frac{n+4}{12}\left[2(a + d_1) + 2\left(\frac{n-2}{6}\right)(d_1 + d_2 + d_3)\right]$$

Case C: When only the terms n = 3, 9, 15, 21, 27, … of the form (6m-3) for m = 1, 2, 3, … are taken into consideration, then we get an arithmetic series with first term $(a+2d_1)$ and common difference $2(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$\mathrm{S_C} = \frac{\mathrm{m}}{2}[2(\mathrm{a}+2\mathrm{d}_1)+(\mathrm{m}-1)2(\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3)]$$
$$= \frac{\mathrm{n}+3}{12}\left[2(\mathrm{a}+2\mathrm{d}_1)+2\left(\frac{\mathrm{n}-3}{6}\right)(\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3)\right]$$

Case D: When only the terms n = 4, 10, 16, 22, 28, … of the form (6m-2) for m = 1, 2, 3, … are taken into consideration, then we get an arithmetic series with first term $(a+2d_1+d_2)$ and common difference $2(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$\mathrm{S_D} = \frac{\mathrm{m}}{2}[2(\mathrm{a}+2\mathrm{d}_1+\mathrm{d}_2)+(\mathrm{m}-1)2(\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3)]$$
$$= \frac{\mathrm{n}+2}{12}\left[2(\mathrm{a}+2\mathrm{d}_1+\mathrm{d}_2)+2\left(\frac{\mathrm{n}-4}{6}\right)(\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3)\right]$$

Case E: When only the terms n = 5, 11, 17, 23, 29, … of the form (6m-1) for m = 1, 2, 3, … are taken into consideration, then we get an arithmetic series with first term $(a+2d_1+2d_2)$ and common difference $2(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$\mathrm{S_E} = \frac{\mathrm{m}}{2}[2(\mathrm{a}+2\mathrm{d}_1+2\mathrm{d}_2)+(\mathrm{m}-1)2(\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3)]$$
$$= \frac{\mathrm{n}+1}{12}\left[2(\mathrm{a}+2\mathrm{d}_1+2\mathrm{d}_2)+2\left(\frac{\mathrm{n}-5}{6}\right)(\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3)\right]$$

Case F: When only the terms n = 6, 12, 18, 24, 30, … of the form (6m) for m = 1, 2, 3, … are taken into consideration, then we get an arithmetic series with first term $(a+2d_1+2d_2+d_3)$ and common difference $2(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$\mathrm{S_F} = \frac{\mathrm{m}}{2}[2(\mathrm{a}+2\mathrm{d}_1+2\mathrm{d}_2+\mathrm{d}_3)+(\mathrm{m}-1)2(\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3)]$$
$$= \frac{\mathrm{n}}{12}\left[2(\mathrm{a}+2\mathrm{d}_1+2\mathrm{d}_2+\mathrm{d}_3)+2\left(\frac{\mathrm{n}-6}{6}\right)(\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3)\right]$$

Therefore the sum of the first n terms of the series of 3-D.A.P.M.(2) is determined as follows by applying the suitable formula depending on the number of terms:

Case I: When n = 1, 7, 13, 19, 25, … of the form (6m-5); where m = 1, 2, 3, … Then S_A has m terms and each from S_B to S_F has (m-1) terms. Therefore

$S_n = S_A + (S_B + S_C + S_D + S_E + S_F)$

$$= \frac{m}{2}[2a+(m-1)2(d_1+d_2+d_3)] + \frac{m-1}{2}[2(5a+9d_1+5d_2+d_3)+10(m-2)(d_1+d_2+d_3)]$$
$$= \frac{n+5}{12}\left[2a+\left(\frac{n-1}{6}\right)2(d_1+d_2+d_3)\right] + \frac{n-1}{12}\left[2(5a+9d_1+5d_2+d_3)+10\left(\frac{n-7}{6}\right)(d_1+d_2+d_3)\right]$$

Case II: When n = 2, 8, 14, 20, 26, … of the form (6m-4); where m = 1, 2, 3, … Then each of S_A and S_B has m terms and each from S_C to S_F has (m-1) terms. Therefore

$S_n = (S_A + S_B) + (S_C + S_D + S_E + S_F)$

$$= \frac{m}{2}[2(2a+d_1)+4(m-1)(d_1+d_2+d_3)]$$
$$+ \frac{m-1}{2}[2(4a+8d_1+5d_2+d_3)+8(m-2)(d_1+d_2+d_3)]$$
$$= \frac{n+4}{12}\left[2(2a+d_1)+4\left(\frac{n-2}{6}\right)(d_1+d_2+d_3)\right]$$
$$+ \frac{n-2}{12}\left[2(4a+8d_1+5d_2+d_3)+8\left(\frac{n-8}{6}\right)(d_1+d_2+d_3)\right]$$

Case III: When n = 3, 9, 15, 21, 27, … of the form (6m-3); where m = 1, 2, 3, … Then each of S_A, S_B, and S_C have m terms and each from S_D to S_F has (m-1) terms. Therefore

$S_n = (S_A + S_B + S_C) + (S_D + S_E + S_F)$

$$= \frac{m}{2}[2(3a+3d_1)+6(m-1)(d_1+d_2+d_3)]$$
$$+ \frac{m-1}{2}[2(3a+6d_1+5d_2+d_3)+6(m-2)(d_1+d_2+d_3)]$$
$$= \frac{n+3}{12}\left[2(3a+3d_1)+6\left(\frac{n-3}{6}\right)(d_1+d_2+d_3)\right]$$
$$+ \frac{n-3}{12}\left[2(3a+6d_1+5d_2+d_3)+6\left(\frac{n-9}{6}\right)(d_1+d_2+d_3)\right]$$

Case IV: When n = 4, 10, 16, 22, 28, … of the form (6m-2); where m = 1, 2, 3, … Then each of S_A, S_B, S_C, and S_D have m terms and each from S_E to S_F has (m-1) terms. Therefore

$S_n = (S_A + S_B + S_C + S_D) + (S_E + S_F)$

$$= \frac{m}{2}[2(4a + 5d_1 + d_2) + 8(m-1)(d_1 + d_2 + d_3)]$$

$$+ \frac{m-1}{2}[2(2a + 4d_1 + 4d_2 + d_3) + 4(m-2)(d_1 + d_2 + d_3)]$$

$$= \frac{n+2}{12}\left[2(4a + 5d_1 + d_2) + 8\left(\frac{n-4}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$+ \frac{n-4}{12}\left[2(2a + 4d_1 + 4d_2 + d_3) + 4\left(\frac{n-10}{6}\right)(d_1 + d_2 + d_3)\right]$$

Case V: When n = 5, 11, 17, 23, 29, … of the form (6m-1); where m = 1, 2, 3, … Then each of S_A, S_B, S_C, S_D, and S_E have m terms and S_F has (m-1) terms. Therefore

$S_n = (S_A + S_B + S_C + S_D + S_E) + S_F$

$$= \frac{m}{2}[2(5a + 7d_1 + 3d_2) + 10(m-1)(d_1 + d_2 + d_3)]$$

$$+ \frac{m-1}{2}[2(a + 2d_1 + 2d_2 + d_3) + 2(m-2)(d_1 + d_2 + d_3)]$$

$$= \frac{n+1}{12}\left[2(5a + 7d_1 + 3d_2) + 10\left(\frac{n-5}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$+ \frac{n-5}{12}\left[2(a + 2d_1 + 2d_2 + d_3) + 2\left(\frac{n-11}{6}\right)(d_1 + d_2 + d_3)\right]$$

Case VI: When n = 6, 12, 18, 24, 30, … of the form (6m); where m = 1, 2, 3, … Then each of S_A, S_B, S_C, S_D, S_E and S_F has m terms. Therefore

$S_n = S_A + S_B + S_C + S_D + S_E + S_F$

$$= \frac{m}{2}[2(6a + 9d_1 + 5d_2 + d_3) + 12(m-1)(d_1 + d_2 + d_3)]$$

$$= \frac{n}{12}\left[2(6a + 9d_1 + 5d_2 + d_3) + 12\left(\frac{n-6}{6}\right)(d_1 + d_2 + d_3)\right]$$

Example 3.3: Find the sum of first 13, 14, 15, 16, 17, and 18 terms separately of a 3-D.A.P.M.(2) series 2 + 5 + 8 + 12 + 16 + 21 + 26 + 29 + 32 + 36 + 40 + …

Here a = 2, $d_1 = 3$, $d_2 = 4$, $d_3 = 5$, M = 2.

For n = 13, we have n = 13 = 6.m – 5 ($m = 3$). Therefore

$$S_{13} = \frac{n+5}{12}\left[2a + \left(\frac{n-1}{6}\right)2(d_1 + d_2 + d_3)\right]$$

$$+ \frac{n-1}{12}\left[2(5a + 9d_1 + 5d_2 + d_3) + 10\left(\frac{n-7}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$= \frac{13+5}{12}\left[4 + \left(\frac{13-1}{6}\right)24\right] + \frac{13-1}{12}\left[2(10 + 27 + 20 + 5) + 10\left(\frac{13-7}{6}\right)12\right]$$

$$= \frac{18}{12}[4 + 48] + [124 + 120] = 322$$

For n = 14, we have n = 14 = 6.m – 4 ($m = 3$). Therefore

$$S_{14} = \frac{n+4}{12}\left[2(2a + d_1) + 4\left(\frac{n-2}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$+ \frac{n-2}{12}\left[2(4a + 8d_1 + 5d_2 + d_3) + 8\left(\frac{n-8}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$= \frac{14+4}{12}\left[2(4+3) + 4\left(\frac{14-2}{6}\right)(3 + 4 + 5)\right] + \frac{14-2}{12}\left[2(8 + 24 + 20 + 5) + 8\left(\frac{14-8}{6}\right)(3 + 4 + 5)\right]$$

$$= \frac{18}{12}[14 + 96] + [114 + 96] = 375$$

For n = 15, we have n = 15 = 6.m – 3 ($m = 3$). Therefore

$$S_{15} = \frac{n+3}{12}\left[2(3a + 3d_1) + 6\left(\frac{n-3}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$+ \frac{n-3}{12}\left[2(3a + 6d_1 + 5d_2 + d_3) + 6\left(\frac{n-9}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$= \frac{15+3}{12}\left[2(6+9) + 6\left(\frac{15-3}{6}\right)12\right] + \frac{15-3}{12}\left[2(6 + 18 + 20 + 5) + 6\left(\frac{15-9}{6}\right)12\right]$$

$$= \frac{18}{12}[30 + 144] + [98 + 72]$$

$$= 431$$

For n = 16, we have n = 16 = 6.m – 2 $(m = 3)$.
Therefore

$$S_{16} = \frac{n+2}{12}\left[2(4a + 5d_1 + d_2) + 8\left(\frac{n-4}{6}\right)(d_1 + d_2 + d_3)\right] + \frac{n-4}{12}\left[2(2a + 4d_1 + 4d_2 + d_3) + 4\left(\frac{n-10}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$= \frac{16+2}{12}\left[2(8 + 15 + 4) + 8\left(\frac{16-4}{6}\right)12\right] + \frac{16-4}{12}\left[2(4 + 12 + 16 + 5) + 4\left(\frac{16-10}{6}\right)12\right]$$

$$= \frac{18}{12}[54 + 192] + [74 + 48] = 491$$

For n = 17, we have n = 17 = 6.m – 1 $(m = 3)$. Therefore

$$S_{17} = \frac{n+1}{12}\left[2(5a + 7d_1 + 3d_2) + 10\left(\frac{n-5}{6}\right)(d_1 + d_2 + d_3)\right] + \frac{n-5}{12}\left[2(a + 2d_1 + 2d_2 + d_3) + 2\left(\frac{n-11}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$= \frac{18}{12}\left[2(10 + 21 + 12) + 10\left(\frac{12}{6}\right)12\right] + \frac{12}{12}\left[2(2 + 6 + 8 + 5) + 2\left(\frac{6}{6}\right)12\right]$$

$$= \frac{18}{12}[86 + 240] + [42 + 24]$$

$$= 555$$

For n = 18, we have n = 18 = 6.m (m = 3). Therefore

$$S_{18} = \frac{n}{12}\left[2(6a + 9d_1 + 5d_2 + d_3) + 12\left(\frac{n-6}{6}\right)(d_1 + d_2 + d_3)\right]$$

$$= \frac{18}{12}\left[2(12 + 27 + 20 + 5) + 12\left(\frac{12}{6}\right)12\right]$$

$$= \frac{18}{12}[128 + 288] = 624$$

3.4. To Insert n Terms Between First and Last Term of a 3–D.A.P.M.(2): Given the first term a and the last term b of a 3–D.A.P. with multiplicity two, if there exists exactly n terms between them, then to write these terms we proceed as follows:
We have given that a = first term (t_1) and b = (n+2)th term (t_{n+2}). Let d_1, d_2, and d_3 be three successive common differences. Then there arise six cases:
Case I: When (n + 2) is of the form (6m – 5; m = 1, 2, …), then we have

$$t_{n+2} = b = a + 2\frac{(n+2)-1}{6}(d_1 + d_2 + d_3)$$

$$= a + 2\frac{n+1}{6}(d_1 + d_2 + d_3)$$

$$\Rightarrow (d_1 + d_2 + d_3) = \frac{3(b-a)}{n+1}$$

From this it is clear that to find d_1, d_2, and d_3, at least two of them must be given. After finding common differences we can write n terms between a and b.
Case II: When (n + 2) is of the form (6m – 4; m = 1, 2, …), then we have

$$t_{n+2} = b = (a + d_1) + 2\frac{(n+2)-2}{6}(d_1 + d_2 + d_3)$$

$$= (a + d_1) + 2\frac{n}{6}(d_1 + d_2 + d_3)$$

$$\Rightarrow (d_1 + d_2 + d_3) = \frac{3(b - a - d_1)}{n}$$

From this it is clear that to find d_1, d_2, and d_3, at least two of them must be given. After finding common differences we can write n terms between a and b.
Case III: When (n + 2) is of the form (6m – 3; m = 1, 2, …), then we have

$$t_{n+2} = b = (a + 2d_1) + 2\frac{(n+2)-3}{6}(d_1 + d_2 + d_3)$$

$$= (a + 2d_1) + 2\frac{n-1}{6}(d_1 + d_2 + d_3)$$

$$\Rightarrow (d_1 + d_2 + d_3) = \frac{3(b - a - 2d_1)}{n-1}$$

From this it is clear that to find d_1, d_2, and d_3, at least two of them must be given. After finding common differences we can write n terms between a and b.

Case IV: When (n + 2) is of the form (6m – 2; m = 1, 2, …), then we have

$$t_{n+2} = b = (a + 2d_1 + d_2) + 2\frac{(n+2)-4}{6}(d_1 + d_2 + d_3)$$
$$= (a + 2d_1 + d_2) + 2\frac{n-2}{6}(d_1 + d_2 + d_3)$$
$$\Rightarrow (d_1 + d_2 + d_3) = \frac{3(b - a - 2d_1 - d_2)}{n-2}$$

From this it is clear that to find d_1, d_2, and d_3, at least two of them must be given. After finding common differences we can write n terms between a and b.

Case V: When (n + 2) is of the form (6m – 1; m = 1, 2, …), then we have

$$t_{n+2} = b = (a + 2d_1 + 2d_2) + 2\frac{(n+2)-5}{6}(d_1 + d_2 + d_3)$$
$$= (a + 2d_1 + 2d_2) + 2\frac{n-3}{6}(d_1 + d_2 + d_3)$$
$$\Rightarrow (d_1 + d_2 + d_3) = \frac{3(b - a - 2d_1 - 2d_2)}{n-3}$$

From this it is clear that to find d_1, d_2, and d_3, at least two of them must be given. After finding common differences we can write n terms between a and b.

Case VI: When (n + 2) is of the form (6m; m = 1, 2, …), then we have

$$t_{n+2} = b = (a + 2d_1 + 2d_2 + d_3) + 2\frac{(n+2)-6}{6}(d_1 + d_2 + d_3)$$
$$= (a + 2d_1 + 2d_2 + d_3) + 2\frac{n-4}{6}(d_1 + d_2 + d_3)$$
$$\Rightarrow (d_1 + d_2 + d_3) = \frac{3(b - a - 2d_1 - 2d_2 - d_3)}{n-4}$$

From this it is clear that to find d_1, d_2, and d_3, at least two of them must be given. After finding common differences we can write n terms between a and b.

Example 3.4: To insert ten terms between the first term 3 and the last term 45 of a 3–D.A.P. with multiplicity two with first c.d. 2 and third c.d. 6.

Here a = 3, b = 45, $d_1 = 2$, $d_3 = 6$ and n + 2 = 12 = 6.m for m = 2 i.e., n = 10. Therefore from the relation

$$(d_1 + d_2 + d_3) = \frac{3(b - a - 2d_1 - 2d_2 - d_3)}{n-4}$$

We get

$$(2 + d_2 + 6) = \frac{3(45 - 3 - 4 - 2d_2 - 6)}{10-4}$$
$$\Rightarrow (d_2 + 8) = \frac{3(32 - 2d_2)}{6} \Rightarrow d_2 = 4$$

Therefore the 3-D.A.P.M.(2) series is 3, 5, 7, 11, 15, 21, 27, 29, 31, 35, 39, 45.

3.5. Arithmetic Means Between Two Consecutive terms of a 3-D.A.P.M.(2): Before finding the arithmetic mean (A.M.), we express the number of terms in the form 6m-5, 6m-4, 6m-3, 6m-2, 6m-1, 6m, 6m+1 and then apply the following formulae according to the terms:

(i) Between t_{6m-5} and t_{6m-4}

$$\mathrm{A.M.} = a + \frac{d_1}{2} + 2(m-1)(d_1 + d_2 + d_3)$$

(ii) Between t_{6m-4} and t_{6m-3}

$$\mathrm{A.M.} = a + \frac{3d_1}{2} + 2(m-1)(d_1 + d_2 + d_3)$$

(iii) Between t_{6m-3} and t_{6m-2}

$$\mathrm{A.M.} = a + 2d_1 + \frac{d_2}{2} + 2(m-1)(d_1 + d_2 + d_3)$$

(iv) Between t_{6m-2} and t_{6m-1}

$$\mathrm{A.M.} = a + 2d_1 + \frac{3d_2}{2} + 2(m-1)(d_1 + d_2 + d_3)$$

(v) Between t_{6m-1} and t_{6m}

$$\mathrm{A.M.} = a + 2d_1 + 2d_2 + \frac{d_3}{2} + 2(m-1)(d_1 + d_2 + d_3)$$

(vi) Between t_{6m} and t_{6m+1}

$$\mathrm{A.M.} = a + d_1 + d_2 + \frac{d_3}{2} + (2m - 1)(d_1 + d_2 + d_3)$$

Example 3.5: Find the A.M. between 6th and 7th terms of 3–D.A.P. with multiplicity two given by 2, 5, 8, 12, 16, 21, 26, 29, 32, 36, 40, 45, 50, 53, … with 2 as first term and 3, 4, 5 as successive common differences.
Here a = 2, $d_1 = 3$, $d_2 = 4$ and $d_3 = 5$. Hence A.M. between 6th and 7th terms is = A.M. between t_{6m} and t_{6m+1} for m = 1 and is given by the formula

$$\mathrm{A.M.} = a + d_1 + d_2 + \frac{d_3}{2} + (2m - 1)(d_1 + d_2 + d_3)$$
$$= 2 + 3 + 4 + \frac{5}{2} + (2.1 - 1)(3 + 4 + 5)$$
$$= 9 + 2.5 + 12 = 23.5$$

3.6. Forming a 3–D.A.P. with Multiplicity Four: By inserting one arithmetic mean between every two consecutive terms in the 3–D.A.P.M.(2), we get 3–D.A.P. with Multiplicity Four as shown below:

$$a$$
$$a + \frac{d_1}{2}$$
$$a + d_1$$
$$a + 3\frac{d_1}{2}$$
$$a + 2d_1$$
$$a + 2d_1 + \frac{d_2}{2}$$
$$a + 2d_1 + d_2$$
$$a + 2d_1 + 3\frac{d_2}{2}$$
$$a + 2d_1 + 2d_2$$
$$a + 2d_1 + 2d_2 + \frac{d_3}{2}$$
$$a + 2d_1 + 2d_2 + d_3$$
$$a + 2d_1 + 2d_2 + 3\frac{d_3}{2}$$
$$a + 2d_1 + 2d_2 + 2d_3$$
$$a + 5\frac{d_1}{2} + 2d_2 + 2d_3$$
$$a + 3d_1 + 2d_2 + 2d_3$$
$$a + 7\frac{d_1}{2} + 2d_2 + 2d_3$$
$$a + 4d_1 + 2d_2 + 2d_3$$
$$a + 4d_1 + 5\frac{d_2}{2} + 2d_3$$

and so on.

3.7. Properties of 3-D.A.P.M.(2): Based on the previous properties on arithmetic progressions and a basic logic, we can easily deduce that

i. If to each terms of a 3-D.A.P.M.(2) a finite number is added or subtracted, the resulting sequence will also be a 3-D.A.P.M.(2).
ii. If each term of a 3-D.A.P.M.(2) is multiplied or divided by a non-zero finite number, the resulting sequence will also be a 3-D.A.P.M.(2).
iii. If corresponding terms of two different 3-D.A.P.M.(2)'s be added or subtracted, the resulting sequence will also be a 3-D.A.P.M.(2).

iv. If corresponding terms of two different 3-D.A.P.M.(2)'s be multiplied or divided , the resulting sequence need not form a 3-D.A.P.M.(2).

v. If we take the three c.d.'s equal, we get an A.P. and all the properties of A.P. from the properties of 3-D.A.P.M.(2).

IV. Conclusion

All the properties developed for arithmetic progressions of dimension one and multiplicity one can be extended for higher dimension and multiplicities. But the limitation of this study is that we cannot write the general terms, sum of the series and arithmetic means in a single form. We have to write them for different cases depending on the dimension and multiplicity.

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